RIEMANN-STIELTJES INTEGRATION

A project submitted

by

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DEPARTMENT OF MATHEMATICS

NILAMANI MAHAVIDYALAYA,

BALASORE, ODISHA-756028

DEDICATED TO

MY BELOVED PARENTS

PROJECT CERTIFICATE

This is to certify that the project entitled **RIEMANN-STIELTJES INTEGRATION** submitted by **Madhaba Ranjan Sahoo** to Nilamani mahavidyalaya, Rupsa for the partial fulfilment of the requirements of B.Sc degree in Mathematics is a bonafides record of review work carried out by him under my supervision and guidance.

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DECLARATION

I hereby declare that the work on the topic **RIEMANN-STIELTJES INTEGRATION** for my B.Sc. degree has been carried out by me in the Department of Mathematics, Nilamani Mahavidyalaya, Rupsa and further declare that it has not been submitted earlier wholly or in part to any other Institution or University for the award of any other degree or diploma.

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RIEMANN-STIELTJES INTEGRATION

0.1 Introduction

In mathematics, the RiemannStieltjes integral is a generalization of the Riemann integral , named after Bernhard Riemann and Thomas Joannes Stieltjes. The definition of this integral was first published in 1894 by Stieltjes . It serves as an instructive and useful precursor of the Lebesgue integral , and an invaluable tool in unifying equivalent forms of statistical theorems that apply to discrete and continuous probability .

0.2 Preliminaries

Definition 0.2.1. Let [a,b] be a given interval by a partition P of [a,b] we mean a finite set of point $x_0, x_1, ..., x_n$ where

$$a = x_0 \leq x_1 \leq .... \leq x_{n-1} \leq x_n = b$$

we write

$$\Delta x_i = x_i - x_i - 1. \quad \text{(i=1,...,n)}$$

now suppose f is a bounded real function defined on [a,b] . corresponding to each partition P of [a,b] me put

$$M_I = supf(x) \quad (x_{i-1} \leq x \leq x_i),$$

$$m_i = inf f(x) \quad (x_{i-1} \leq x \leq x_i),$$

$$U(p, f) = \sum_{i=1}^{n} M_i \Delta x_i,$$

$$L(p, f) = \sum_{i=1}^{n} m_i \Delta x_i,$$

and finally

$$\int_a^{-b} f dx = inf\ U(P, f) \tag{0.1}$$

$$\int_{-a}^{b} f dx = inf\ L(P, f) \tag{0.2}$$

where the inf and the sup are taken over all partitions P of [a,b] . the left member of (0.1) and (0 .2) are called the upper and lower riemann integral of f over [a,b] , respectively
if the upper and lower integrals are equal , we say that f is riemann-integrable on [a,b] , we write f $\in R$ (that is , R denotes the set of riemannintegrable function) , and we denote the common valu of (0.1) and (0.2) by

$$\int_a^b \tag{0.3}$$

or by

$$\int_a^b f dx \tag{0.4}$$

This is the riemann integral of f over [a,b] . since f is bounded , there exist two number , m and M , such that

$$m \leq f(x) \leq M \quad (a \leq x \leq b).$$

Hence , for every P

$$m(b-a) \leq L(p,f) \leq U(p,f) \leq M(b-a) ,$$

so that the number L(p,f) and U(p,f) form bounded set . This shows that the upper and lower integrals are defined for every bounded function f . The question of their equality , and hence the question of the integrability of f , is a more delicate one . Instead of investigating it separately for the Riemann integral , we shall immediately consider a more general situation

Definition 0.2.2. Let a be a monotonically increasing function on [a,b] (since$\alpha(a)$ *and* $\alpha(b)$ are finite , it follows that α is bounded on $[a,b]$) . Corresponding to each partition P of [a,b,] we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_i - 1).$$

it is clear that $\Delta\alpha_i \geq 0$. for any real function f which is bounded on [a,b] we put

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta x_i ,$$

$$L(P,f,\alpha)=\sum_{i=1}^{n} M_i\Delta x_i \ ,$$

where M_i, m_i have the same meaning as in defination 2.1, and we define

$$\int_a^{-b} f dx = inf\ U(p,f,\alpha), \tag{0.5}$$

$$\int_{-a}^{b} f dx = sup\ L(p,f,\alpha), \tag{0.6}$$

the inf and sup again being taken over all partitions .

if the left member of (0.5) and (0.6) are equal , we denote their common valu by

$$\int_a^b f dx \tag{0.7}$$

or sometimes by

$$\int_a^b f dx(x) \tag{0.8}$$

this is the Riemann-stieltijs integral (or simply the stieltjes integral) of $f$with respect to α , over [a,b]
if (0.7) exists i.e , if (0.5) and (0.6) are equal , we say that f is integrable with respect to α , in the Riemann sence , and write $f \in R(\alpha)$.
By taking $\alpha(x) = x$, the riemann integral is seen to be a special case of the riemann-stieltjes integral . let us mention explicity , however that in the general case α need not even be continuous .
A few word should be said about the notation . we prefer (0.7) to (0.8) , since the letter (α) which appers in (0.8) adds nothing to the content of (0.7) . it is immaterial which letter we use to represent the so called $(''variable of integration'')$. for instance , (0.8) is the same as

$$\int_a^b f(y) d\alpha(y).$$

The integral depends on f, α, a and b , but not on the variable of integration , which may as well be omitted .

The role played by the variable of integration is quite analougs to that of the index of summation :the symbols

$$\sum_{i=1}^{n} c_i, \quad \sum_{k=1}^{n} c_k$$

mean the same thing , since each means $c_1 + c_2 + ... + c_n$.

Of course , no harm is done by inserting the variable of integration , and in many case it is actually convenient to do so .

We shall now investigate the existance of the integral (0.7). without saying so every time f will be assumed real and bounded , and α monotonically increasing on $[a, b]$;and ,when there can be no misunderstanding ,we shall write $\int$ in place of $\int_a^b$.

Definition 0.2.3. We say that the partition(P^* is a refinement of P if $P^* \supset P$ (that is, if every point of P is a point of P^*). Given two partitions,P_1 and P_2, we say that P^* is their common refinement if $P^* = P_1 \cup P_2$

Theorem 0.2.4. *if P^* is a refinement of P, then*

$$L(P, f, \alpha) \leq L(P^*, f, \alpha) \tag{0.9}$$

and

$$U(P,^* f, \alpha) \leq L(P, f, \alpha). \tag{0.10}$$

Proof. To prove (0.9), suppose first that P^* contains just one point more than P . Let this extra point be x^* and suppose $x_{i-1} < x^* < x_i$, where $x_{i-1} and x_i$ are two consecutive points of P. Put

$$W_1 = inf f(x)(x_{i-1} \leq x \leq x^*),$$

$$W_2 = inf f(x)(x_* \leq x \leq x_i),$$

Clearly $w_1 \geq m_i and w_2 \geq m_i$, where as before ,

$$m_i = inf f(x)(x_{i-1} \leq x \leq x_i),$$

Hence

$L(P^*, f, \alpha) - L(P, F, \alpha)$

$$= W_1[\alpha(x^*) - \alpha(x_{i-1})\ ] + W_2[\alpha(x_i) - \alpha(x^*)] - m_i[\alpha(x_i) - \alpha(x_{i-1})]$$

$$= (W_1 - m_i)[\alpha(x^* - \alpha(x_{i-1})] + (W_2 - m_i)[\alpha(x_i) - \alpha(x^*)] \geq 0.$$

If P^* contains K points more than P , we repeat this reasoning K times, and arrive at (0.9). The proof of (0.10) is analogous.

□

Theorem 0.2.5. $\int_{-a}^{b} f dx \leq \int_{a}^{-b} f dx$

Proof. Let P^* be the common refinement of two partitions P_1 and P_2. by theorem 2.4

$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha)$.

Hence

$$L(P_1, f, \alpha,) \leqslant U(P_2, f, \alpha) \tag{0.11}$$

If P_2 is fixed and the sup is taken over all P_1,(0.11) gives

$$\underline{\int} f dx \leq U(P_2, f, \alpha). \tag{0.12}$$

The theorem follows by taking the inf over all $P_2 in$(0.12). □

Theorem 0.2.6. *$f \in R(\alpha)$ on [a, b] if and only if for every $\in > 0$ there exists a partition P such that*

$$U(P, f, \alpha) - L(P, f, \alpha) < \in . \tag{0.13}$$

Proof. For every P we have

$$L(P, f, \alpha) \leq \underline{\int} f dx \leq \overline{\int} f dx \leq U(P, f, \alpha.)$$

Thus(0.13) implies

$$0 \leq \overline{\int} f dx - \underline{\int} f dx < \in .$$

Hence, if (0.13) can be satisfied for every $\in> 0$, we have

$$\int^{-} f dx = \int_{-} f dx,$$

That is, $f \in R(\alpha)$

Conversely, suppose $f \in R(\alpha)$, and let $\in> 0$ be given. Then there exist partitions P_1 and P_2 such that

$$U(P_2, f, \alpha) - \int f dx < \frac{\in}{2}, \tag{0.14}$$

$$\int f dx - L(P_1, f, \alpha) < \frac{\in}{2}, \tag{0.15}$$

We choose P to be the common refinement of P_1 and P_2. Then Theorem 2.4, together with(0.14)and(0.15), shows that

$$U(P, f, \alpha) \leqslant U(P_2, f, \alpha) < \int f dx + \frac{\in}{2} < L(P_1, f, \alpha) + \in \leqslant L(P, f, \alpha) + \in,$$

so that (0.13) holds for this partition P .

Theorem 2.6 furnishes a convenient criterion for integrability . before we apply it , we state some closely related facts . □

Theorem 0.2.7. *(a) If (0.13) holds for some P and some ϵ, then (0.13) holds (with the same ϵ) for every refinement of P.*

(b) If (0.13) holds for $P = (x_0, ..., x_n)$ and if S_i , $t_i$are arbitrary points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} | f(s_i) - f(t_i) | \Delta x_i < \in$$

(c) If $f \in R$ $(\alpha)$and the hypotheses of (b) hold, then b

$$| \sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f dx | < \epsilon.$$

Proof. Theorem 2.4 implies (a). Under the assumptions made in (b), both $f(s_i)$ and $f(t_i)$ lie in$[m_i, M_i]$, so that $\mid f(s_i - f(t_i) \mid\leqslant Mi - m_i$. Thus

$$\sum_{i=1}^{n} \mid f(s_i) - f(t_i) \mid \Delta_i \leqslant U(P, f, \alpha) - L(P, f, \alpha)$$

which proves (b). The obvious inequalities

$$L(P, f, \alpha) \leqslant \sum f(t_i)\Delta\alpha_i \leqslant U((P, f, \alpha)$$

and

$$L(P, f, \alpha) \leqslant \int f dx \leqslant U(P, f, \alpha)$$

prove (c). □

Theorem 0.2.8. *If f is continuous on [a, b] then* $f\epsilon R(\alpha)$ *on* $[a, b]$.

Proof. Let $\epsilon > 0$ be given. Choose $\eta > 0$ so that

$$[\alpha(b) - \alpha(a)]\eta < \epsilon.$$

Since f is uniformly continuous on $[a, b]$, there exists a $\delta > 0$ such that

$$\mid f(x) - f(t) \mid< \eta \tag{0.16}$$

if $x \in [a, b]$, t $\in$[a, b], and $\mid x - t \mid< \delta$
. If P is any partition of $[a, b]$ such that $\Delta x_i < \delta$ for all i, then (0.16) implies that

$$(M_i - m_i \leq \eta \quad (i - 1, ..., n)$$

and therefore

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i$$

$$\leqslant \eta \sum_{i=1}^{n} \Delta x_i = \eta[\alpha(b) - \alpha(a)] < \in .$$

By Theorem 2.6, $f \in R(\alpha)$.

□

Theorem 0.2.9. *If f is monotonic $[a, b]$, and if α is continuous on $[a, b]$, then $f \in R(\alpha)$. (We still assume, of course, that α is monotonic).*

Proof. Let $\in > 0$ be given. For any positive integer n, choose a partition such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \qquad (i = 1, ...n).$$

This is possible since α is continuous .
We suppose that f is monotonically increasing (the proof is analogous in the other case). Then

$$M_i = f(x_i), \qquad m_i = f(x_{i-1}) \qquad (i = 1, ..., n)$$

so that

$$U(P, f, \alpha) - L(P, f, \alpha) = \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} . [f(b) - f(a)] < \epsilon$$

if n is taken large enough. By Theorem 2.6, $f \epsilon R(\alpha)$ □

Theorem 0.2.10. *Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $f \in R(\alpha)$.*

Proof. Let $\in > 0$ be given. Put M = sup $| f((x) |$let E be the set of points at which f is discontinuous. Since E is finite and α is continuous at every point of E, we can cover E by finitely many disjoint intervals $[u_j, v_j] \subset$ [a, b] such that the sum of the corresponding differences $\alpha(vj) - \alpha(uj)$ is less than

$$\leqslant \eta \sum_{i=1}^{n} \Delta x_i = \eta[\alpha(b) - \alpha(a)] < \in .$$

By Theorem 2.6, $f \in R(\alpha)$.

□

Theorem 0.2.9. *If f is monotonic $[a, b]$, and if α is continuous on $[a, b]$, then $f \in R(\alpha)$. (We still assume, of course, that α is monotonic).*

Proof. Let $\in > 0$ be given. For any positive integer n, choose a partition such that

$$\Delta \alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \qquad (i = 1, ...n).$$

This is possible since α is continuous .
We suppose that f is monotonically increasing (the proof is analogous in the other case). Then

$$M_i = f(x_i), \qquad m_i = f(x_{i-1}) \qquad (i = 1, ..., n)$$

so that

$$U(P, f, \alpha) - L(P, f, \alpha) = \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} . [f(b) - f(a)] < \epsilon$$

if n is taken large enough. By Theorem 2.6, $f \epsilon R(\alpha)$ □

Theorem 0.2.10. *Suppose $f$is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $f \in R(\alpha)$.*

Proof. Let $\in > 0$ be given. Put M = sup $\mid f((x) \mid$let E be the set of points at which f is discontinuous. Since E is finite and α is continuous at every point of E, we can cover E by finitely many disjoint intervals $[u_j, v_j] \subset$ [a, b] such that the sum of the corresponding differences $\alpha(vj) - \alpha(uj)$ is less than

$\in$. Furthermore, we can place these intervals in such a way that every point of $E \cap (a, b)$ lies in the interior of some $[u_j, v_j]$. Remove the segments (u_j, v_j) from $[a, b]$. The remaining set K is compact. Hence f is uniformly continuous on K, and there exists $\delta > 0$ such that $\mid f(s) - f(t) < \in$ if $s \in K, \mid s - t \mid < \delta$
Now form a partition $P = (x_0, x_1 ... x_n)$ of [a, b], as follows : Each u_j occurs in P. Each v_j occurs in P. No point of any segment(u_j, v_j) occurs in P. If x_{i-1} is not one of theu_j , then $\Delta x_i < \delta$
Note that$M_i - m_i \leqslant 2M$ for every i, and that$M_i - m_i \leqslant \in$e unless X_{i-1} is one of the u_j Hence, as in the proof of Theorem 2.8,

$$U(P, f, \alpha) - L(P, f, \alpha) \leqslant [\alpha(b) - \alpha(a)] \in +2M \in .$$

Since $\in$ is arbitrary, Theorem 2.6 shows that $f \in R\alpha$.
Note: If f and α have a common point of discontinuity, then f need not be in $R(\alpha)$. shows this.

□

Theorem 0.2.11. *Suppose $f \in R(\alpha)$ on $[a, b]$, $m \leqslant f \leqslant M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in R(\alpha)$ on $[a, b]$.*

Proof. Choose $\in > 0$. Since ϕ is uniformly continuous on$[m, M]$,there exists $\delta > 0$ such that $\delta < \in$ and $\mid \phi(s) - \phi(t) \mid < \in$ if $\mid s - t \mid \leqslant \delta$and s,t$\in [m, M]$.
Since$f \in R(\alpha)$, there is a partition $P = (x_0, x_1, ..., x_n)$of$[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$.

LetM_i, m_i have the same meaning as in Definition 2. 1 , and letM_i^*, m_i^* be the analogous numbers for$h$. Divide the numbers $1, ..., n$ into two classes :$i \in (A)$ if $M_i - m_i < \delta$, $i \in B$ if $M_i - m_i \geqslant \delta$.

$$For \quad i \in A, our choice of \delta shows that M_i^* - m_i^* \leqslant \in e.$$

$$For \quad i \in B, M_i^* - m_i^* \leqslant 2K, where K = sup \mid \phi(t) \mid m \leq t \leq M. by$$

(0.16) we have

$$\delta \sum_{i \in b} \Delta\alpha_i \leqslant \sum_{i \in b} (M_i - m_i)\Delta\alpha_i < \delta^2 \qquad (0.17)$$

so that $\sum_{i\in B}\Delta\alpha_i < \delta$. it follows that
$U(P,h,\alpha) - L(P,h,\alpha) = \sum_{i\in A}(M_i^* - m_i^*)\Delta\alpha_i + \sum_{i\in B}(M_i^* - m_i^*)\Delta\alpha_i$

$$\leqslant \in [\alpha(b) - \alpha(a)] + 2K\delta < \in [\alpha(b) - \alpha(a) + 2K].$$

Since $\in$ was arbitrary, Theorem 2.6 implies that $h \in R(\alpha)$.
Remark: This theorem suggests the question : Just what functions are Riemann-integrable ? The answer is given by . □

0.3 Properties of the integral

Theorem 0.3.1. *(a) If $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$ on [a,b] ,then*

$$f_1 + f_2 \in R(\alpha)$$
$$cf \in R(\alpha)$$

for every constant c ,and

$$\int_a^b (f_1 + f_2)dx = \int_a^b f_1 dx + \int_a^b f_2 dx,$$

$$\int_a^b cf dx = c\int_a^b f dx$$

(b) if $f_1(x) \leqslant f_2(x)$ on $[a,b]$, then

$$\int_a^b f_1 dx \leqslant \int_a^b f_2 dx.$$

(c) if $f \in R(\alpha)$ on $[a,b]$ and if $a < c < b$,then $f \in R(\alpha)$ on $[a,c]$ and on $[c,b]$ and

$$\int_a^c f dx + \int_c^b f dx = \int_a^b f dx.$$

(d) if $f \in R(\alpha) on [a,b]$ and if $| f(x) | \leqslant M on [a,b]$, then

$$\left| \int_a^b f dx \right| \leqslant M[\alpha(b) - \alpha(a)].$$

$(e) if f \in R(\alpha_1) and f \in R(\alpha_2), then f \in R(\alpha_1 + \alpha_2)$ *and*

$$\int_a^b fd((\alpha_1) + (\alpha_2) = \int_a^b fdx_1 + \int_a^b fdx_2;$$

$if\ f \in R(\alpha) and \quad c\ is\ a\ positive constant, then f \in R(c\alpha) and$

$$\int_a^b fd(c\alpha) = c\int_a^b fdx.$$

Proof. If $f = f_1 + f_2$ and$P$is any partition of $[a, b]$, we have

$$L(P, f_1, \alpha)+L(P, , f_2, \alpha) \leqslant L(P, f, \alpha) \leq U(P, f, \alpha) \leqslant U(P, f_1, \alpha)+U(P, f_2, \alpha). \tag{0.18}$$

if $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$,let $\in > 0$ be given . There are partitions P_j $(j = 1, 2)$such that

$$U(P_j, f_j, \alpha) - L(P_J, f_j, \alpha) < \in$$

These inequalities persist ifP_1andP_2 are replaced by their common refinement$P$. Then (0.18) implies

$$U(P, f, \alpha) - L(P, f, \alpha) < 2 \in$$

, which proves that $f \in R(\alpha)$.
With this same P we have

$$U(P, f_j, \alpha) < \int f_j dx + \in \qquad (j = 1, 2)$$

hence (0.18) implies

$$\int fdx \leqslant U(P, f, \alpha) < \int f_1 dx + \int f_2 dx + 2 \in$$

Since $\in$ was arbitrary, we conclude that

$$\int fdx \leqslant \int f_1 dx + \int f_2 dx \tag{0.19}$$

if we replace f_1 and f_2 in(21) by$-f_1$ and $-f_2$' the inequality is reversed, and the equality is proved.

The proofs of the other assertions of Theorem 3.1 are so similar that we omit the details. In part (c) the point is that (by passing to refinements) we may restrict ourselves to partitions which contain the point c, in approximating $\int f dx$

□

Theorem 0.3.2. *If $f \in R(\alpha)$ and$g \in R(\alpha$ on [a,b], then*
(a) $f_g \in R(\alpha)$;
(b)$\mid f \mid \in R(\alpha)$and $\mid \int_a^b f dx \mid \leqslant \int_a^b \mid f \mid dx$

Proof. If we take $\phi(t) = t^2$, Theorem 3.1 shows that$f^2 \in R(\alpha)$ if $f \in R(\alpha)$. The identity

$$4fg = (f+g)^2 - (f-g)^2$$

completes the proof of (a).
If we take $\phi(t) = \mid t \mid$, Theorem 3.1 shows similarly that $\mid f \mid \in R(\alpha)$, Choose $c = \pm 1$, so that

$$c \int f dx \geqslant 0$$

Then

$$\mid \int f dx \mid = c \int f dx = \int c f dx \leq \int \mid f \mid dx,$$

since $cf \leq \mid f \mid$.

□

Definition 0.3.3. The unit step function I defined by

$$I(x) = 0 (x \leqslant 0), \quad 1(x > 0)$$

Theorem 0.3.4. *If $a < s < b$, f is bounded on [a, b], f is continuous at s, and $\alpha(x) = I(x-s)$, then*

$$\int_a^b f dx = f(s)$$

Proof. Consider partitions $P = (x_0, x_1 x_2, x_3)$, where $x_0 = a$, and $x_1 = s < x_2 < x_3 = b$ then

$$U(P, f, \alpha) = M_2, \quad L(P, f, \alpha) = m_2.$$

Since f is continuous at s, we see that M_2 and m_2 converge to f(s) as $x_2 \to s$ □

Theorem 0.3.5. *Suppose $C_n \geqslant 0$ for $1, 2, 3, ..., \sum c_n$ converges, s_n is a sequence of distinct points in (a, b), and*
$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$

Let f be continuous on [a, b]. Then

$$\int_a^b f dx = \sum_{n=1}^{\infty} c_n f(s_n). \tag{0.20}$$

Proof. The comparison test shows that the series converges for every x. Its sum $\alpha(x)$ is evidently monotonic, and $\alpha(a) = 0, \alpha(b) = \sum c_n$.(This is the type of function that occurred in Remark .
Let $\in > 0$ be given, and choose N so that

$$\sum_{N+1}^{\infty} c_n < \in$$

Put

$$\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n), \quad \alpha_2(x) = \sum_{N+1}^{\infty} c_n I(x - s_n),$$

By Theorems 3.1 and 3. 4,

$$\int_a^b f dx_1 = \sum_{i=1} c_n f(s_n). \tag{0.21}$$

since $\alpha_2(b) - \alpha_2(a) < \in$,

$$| \int_a^b f dx_2 | \leqslant M_{\in}, \tag{0.22}$$

where$M = sup \mid f(x) \mid$ Since$\alpha = \alpha_1 + \alpha_2$, it follows from (0.24) and (0.25) that

$$\mid \int_a^b f dx - \sum_{i=1}^N c_n f(s_n) \mid \leqslant M\epsilon.$$

If we let $N \to \infty$, we obtain (0.20). □

Theorem 0.3.6. *Assume α increases monotonically and $\alpha' \in R$ on [a, b]. Let f be a bounded real function on [a, b]. Then $f \in R(\alpha)$ if and only iff$\alpha' \in R$ In that case*
$\int_a^b f dx = \int_a^b f(x)\alpha'(x)dx.$

Proof. Let $\epsilon > 0$ be given and apply Theorem 2.6 to α' : There is a partition $P = x_0, ..., x_n of [a, b]$ such that

$$U(P, \alpha') - L(P, \alpha') < \epsilon . \tag{0.23}$$

The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$\Delta\alpha_i = \alpha'(t_i)\Delta x_i$$

for $i = 1, ..., n$. If$S_i \in [x_{i-1}, x_i]$, then

$$\sum_{i=1}^n \mid \alpha'(s_i) - \alpha'(t_i) \mid \Delta x_i < \epsilon \tag{0.24}$$

by (0.23) and Theorem 2.7(b). Put$M = sup \mid f(x) \mid$. Since

$$\sum_{i=1}^n f(s_i)\Delta\alpha_i = \sum_{i=1}^n f(s_i)\alpha'(t_i)\Delta x_i$$

it follows from (0.24) that

$$\mid \sum_{i=1}^n f(s_i)\Delta\alpha_i - \sum_{i=1}^n f(s_i)\alpha'(s_i)\Delta x_i \mid \leqslant M \epsilon . \tag{0.25}$$

In particular,

$$\sum_{i=1}^n f(s_i)\Delta\alpha_i \leqslant U(P, f\alpha') + M \epsilon,$$

for all choices of$S_i \in [[x_{i-1} - x_i]$, so that

$$U(P, f, \alpha) \leqslant U(P, f, \alpha') + M \in .$$

The same argument leads from (0.25) to

$$U((P, f, \alpha') \leqslant U(P, f, \alpha) + M \in$$

. Thus

$$| \, U((P, f, \alpha) - U((P, f, \alpha') \, | \leqslant M \in \tag{0.26}$$

Now note that (0.23) remains true if P is replaced by any refinement. Hence (0.26) also remains true. We conclude that

$$| \int_a^{-b} f dx - \int_a^{-b} f x \alpha'(x) dx \, | \leqslant M \in .$$

But $\in$ is arbitrary. Hence

$\int_a^{-b} f dx = \int_a^{-b} f x \alpha'(x) dx,$

for any bounded f The equality of the lower integrals follows from (0.25) in exactly the same way. The theorem follows. □

Remark The two preceding theorems illustrate the generality and flexibility which are inherent in the Stieltjes process of integration. If α is a pure step function [this is the name often given to functions of the form , the integral reduces to a finite or infinite series. If α has an integrable derivative, the integral reduces to an ordinary Riemann integral. This makes it possible in many cases to study series and integrals simultaneously, rather than separately.

To illustrate this point, consider a physical example. The moment of inertia of a straight wire of unit length, about an axis through an endpoint, at right angles to the wire, is

$$\int_0^1 x^2 dm \tag{0.27}$$

where $m(x)$ is the mass contained in the interval $[0, x]$. If the wire is regarded as having a continuous density p, that is, if m'(x) = p(x), then (0.27) turns into

$$\int_0^1 x^2 p(x) dx \tag{0.28}$$

On the other hand, if the wire is composed of masses m_i concentrated at points x_i (0.27) becomes

$$\sum_i x_i^2 m_i. \tag{0.29}$$

Thus (0.27) contains (0.28) and (0.29) as special cases, but it contains much more ; for instance, the case in which m is continuous but not everywhere differentiable.

Theorem 0.3.7. *(change of variable) Suppose φ is a strictly increasing continuous function that maps an interval [A, B] onto [a, b]. Suppose α is monotonically increasing on [a, b] and $f \in R(\alpha)$ on [a, b]. Define β and g on [A, B] by*

$$\beta(y) = \alpha(\varphi(y)), \qquad g(y) = f(\varphi(y)).$$

Then $g \in R(\beta)$ and $\int_A^B g d\beta = \int_a^b f dx$

Proof. To each partition$P = (x_0, ..., x_n)$ of [a, b] corresponds a partition $Q = (y_0, ..., y_n)$ of [A, B], so that $x_i = \varphi(y_i)$. All partitions of [A, B] are obtained in this way. Since the values taken by f on $[x_{i-1}, x_i]$,are exactly the same as those taken by g on $[y_{i-1}, y_i]$, we see that

$$U(Q, g, \beta) = U(P, f, \alpha), \qquad L(Q, g, \beta) = L(P, f, \alpha) \tag{0.30}$$

Since $f \in R(\alpha)$ P can be chosen so that both $U(P, f, \alpha)$ and$L(P, f, \alpha)$ are close to $\int f dx$ Hence (0.30), combined with Theorem 3.6, shows that

$g \in R(\beta)$ and that holds. This completes the proof.
Let us note the following special case :
Take $\alpha(x) =$ x. Then $\beta = \varphi$. Assume $\varphi' \in R$ on [A, B]. If Theorem 3.6 is applied to the left side of , we obtain

$$\int_a^b f dx = \int_A^B f(\varphi(y))\varphi'(y)dy.$$

□

0.4 *Integration and differentiation*

We still confine ourselves to real functions in this section. We shall show that integration and differentiation are, in a certain sense, inverse operations.

Theorem 0.4.1. *Let* $f \in R$ *on [a, b]. For* $a \leqslant x \leqslant b$, *put*

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on [a, b],. furthermore, if f is continuous at a point x_0 *of [a, b], then F is differentiable at* x_0 , *and*

$$F'(x_0) = f(x_0).$$

Proof. Since $f \in R$ f is bounded Suppose $| f(t) | \leqslant M$ for $a \leqslant t \leqslant b$. if $a \leqslant x < y \leqslant b$, then

$$| F(y) - F(x) | = | \int_x^y f(t)dt | \leqslant M(y - x),$$

by Theorem 3.1(c) and (d). Given $\in > 0$, we see that

$$| F(y) - F(x) | < \in,$$

provided that $| y - x | < \in /M$. This proves continuity (and, in fact, uniform continuity) of F.
Now suppose $f$is continuous at x_0 . Given $\in > 0$, choose $\delta > 0$such that

$$| f(t) - f(x_0) | < \in$$

if $| t - x_0 | < \delta$, *and* $a \leqslant t \leqslant b$. Hence, if

$$x_0 - \delta < s \leqslant x_0 \leqslant t < x_0 + \delta \quad and \quad a \leqslant s < t \leqslant b,$$

we have ,by theorem 3.1 (d) ,
$| \dfrac{F(t) - F(s)}{t - s} - f(x_0) | = | \dfrac{1}{t - s} \int_s^t [f(u) - f(x_0)] du | < \in$.
it follows that $F'(x_0) = f(x_0)$.

□

Theorem 0.4.2. The fundamental theorem of calculus *If* $f \in R$ *on* $[a, b]$ *and if there is a diffe rentiable function* $F on [a, b]$ *such that* $F' = f$, *then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Let $\in > 0$ be given. Choose a partition $P = (x_0, ..., x_n) of [a, b]$ so that $U(P, f) - L(P, f) < \in$. The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$$

for $i = 1, ...n$ thus

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = F(b) - F(a).$$

It now follows from Theorem 3.6(c) that

$$| F(b) - F(a) - \int_a^b f(x)dx | < \in$$

Since this holds for every $\in > 0$, the proof is complete.

□

Theorem 0.4.3. *(Integration by parts) Suppose F and G are differentiable functions on [a, b],* $F' = f \in R$, *and* $G' = g \in R$ *Then*

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

Proof Put $H(x) = F(x)G(x)$ *and apply Theorem 4.2 to H and its derivative. Note that*$H' \in R$,

0.5 *Integration of vector -valued function*

Definition 0.5.1. Let $f_1, ..., f_k$ be real functions on [a, b], and let$f = (f_1..., f_k)$ be the corresponding mapping of [a, b] into R^k. If α increases monotonically on [a, b], to say that$f \in R(\alpha)$ means that $f_j \in R(\alpha)$ for j = 1 , . . . , k. If this is the case, we define

$$\int_a^b f dx = (\int_a^b f_1 dx, ..., \int_a^b f_k dx).$$

In other words, $\int f dx$ is the point in R^k whose jth coordinate is $\int f_j dx$
It is clear that parts (a), (c), and (e) of are valid for these vector-valued integrals ; we simply apply the earlier results to each coordinate. The same is true . To illustrate, we state the analogue .

Theorem 0.5.2. *If f and f map [a, b] into* R^k, *if* $f \in R$ *on [a, b], and if* $F' = f$, *then*

$$\int_a^b f(t)dt = F(b) - F(a).$$

The analogue of Theorem 6.2(b)offers some new features, however, at east in its proof.

Theorem 0.5.3. *if F maps [a, b] into* R^k *and if* $f \in R(\alpha)$ *for some monotonically increasing function* α *on [a, b], then* $| f | \in R(\alpha)$, *and*

$$|\int_a^b f dx| \leqslant \int_a^b |f| dx$$

Proof. If $f_1 ..., f_n$ are the components of f, then

$$|f| = (f_1^2) + ... + f_k^2)^{1/2}. \quad (0.31)$$

By , each of the functions f_i^2 belongs to $R(\alpha)$; hence so does their sum. Since x^2 is a continuous function of x, Theorem 4.6 shows that the square-root function is continuous on [0, M], for every real M. If we apply once more, (0.31) shows that $|f| \in R(\alpha)$.
To prove (40), put $y = (y_1 ..., y_k)$, where $y_J = \int f_j dx$. Then we have $y = \int f dx$ and

$$|y|^2 = \sum y_i^2 = \sum y_j \int f_j dx = \int (\sum y_j f_J) dx.$$

By the Schwarz inequality,

$$\sum y_j f_j(t) \leqslant |y| |f(t)| \ (a \leqslant t \leqslant b);$$

hence Theorem 4.1(b) implies

$$|y|^2 \leqslant |y| \int |f| dxion$$

if $y = 0$, is trivial if y on , division of (by $|y| gives,$ □

0.6 *Rectifiable*

We conclude this chapter with a topic of geometric interest which provides an application of some of the preceding theory. The case$k = 2$(i.e., the case of plane curves) is of considerable importance in the study of analytic functions of a complex variable.

Definition 0.6.1. A continuous mapping y of an interval [a, b] into$R^k$is called a curve inR^k To emphasize the parameter interval [a, b], we may also say that y is a curve on [a, b).
If y is one-to-one, y is called an arc.

If$y(a)$ = y(b), y is said to be a closed curve.
It should be noted that we define a curve to be a mapping, not a point set. Of course, with each curve y inR^kthere is associated a subset ofR^k namely the range of y, but different curves may have the same range.
We associate to each partition $P = x_0, ..., x_n$ of [a, b] and to each curve y on [a, b] the number

$$\wedge(P, y) = \sum_{i=1}^{n} \mid y(x_1) - y(x_{i-1}) \mid,$$

The ith term in this sum is the distance (inR^k) between the points$y(x_{i-1}-$ $l)$ and $y(x_i)$' Hence $\wedge(P, y)$ is the length of a polygonal path with vertices at$y(x_0)$, $y(x_1), ..., y(x_n)$, in this order. As our partition becomes finer and finer, this polygon approaches the range of y more and more closely. This makes it seem reasonable to define the length of y as

$$\wedge(y) = sup \wedge (P, y),$$

where the supremum is taken over all partitions of [a, b].
If $\wedge(y) < \infty$, we say that y is rectifiable.
In certain cases,$\wedge(y)$ is given by a Riemann integral. We shall prove this for continuously diffe rentiable curves, i.e., for curves y whose derivative y' is continuous.

Theorem 0.6.2. *If y' is continuous on [a, b], then y is rectifiable, and*

$$\wedge(y) = \int_a^b \mid y'(t) \mid dt$$

Proof. If a $a \leqslant x_{i-1} < x_i \leqslant b$ then

$$\mid y(x_i) - y(x_{i-1}) \mid = \mid \int_{xi-1}^{x_i} y'(t)dt \mid \leqslant \int_{xi-1}^{x_i} \mid y'(t) \mid dt$$

Hence

$$\wedge(P, y) \leqslant \int_a^b \mid y'(t) \mid dt$$

for every partition P of [a, b]. Consequently,

$$\wedge(y) \leqslant \int_a^b \mid y'(t) \mid dt$$

To prove the opposite inequality, let $\in > 0$ be given. Since y' is uniformly continuous on [a, b], there exists $\delta > 0$ such that

$$\mid y'(s) - \mid y'(t) < \in \quad if \mid s - t \mid < \delta$$

Let $x_0, ..., x_n$ be a partition of [a, b], with $\Delta x_i < \delta$ for all i. If $x_{i-1} \leqslant x_i$ it follows that

$$\mid y'(t) \mid \leqslant \mid y'(x_i) + \in$$

Hence
$\int_{xi-1}^{x_i} \mid y'(t) \mid dt \leqslant \mid y'(x_i) \mid \Delta x_i + \in \Delta x_i$

$$= \mid \int_{xi-1}^{x_i} [y'(t) + y'(x_i) - y'(t)] dt \mid + \in \Delta x_i$$

$$\leqslant \mid \int_{xi-1}^{x_i} y'(t) \mid + \mid \int_{xi-1}^{x_i} [y'(x_i) + y'(t) -] dt \mid + \in \Delta x_i$$

$$\mid y(x_i) - y(x_{i-1}) \mid + 2 \in \Delta x_i,$$

If we add these inequalities, we obtain

$$\int_a^b \mid y'(t) \mid dt \leqslant \wedge(P, y) + 2 \in (b - a)$$

$$\leqslant \wedge(y) + 2 \in (b - a)$$

Since $\in$ was arbitrary,

$$\int_a^b \mid y'(t) \mid dt \leqslant \wedge(y)$$

□

This completes the proof.

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